ON DEPENDENCE AND RELIABILITY COMPUTATIONS.

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Abstract.
When computing the reliability of a system consisting of several components, it is usually assumed that the components are statistically independent of each other. In case the components are associated, it is known that this leads to underestimation if the system is series, whereas the converse holds for parallel systems. In this paper we consider general monotone systems and study the error resulting from the independence assumption when the component states are in fact distributed according to certain dependence models. We also consider some applications of the results to network systems.

SYSTEM RELIABILITY; BINARY DEPENDENCE MODELS; ERROR ESTIMATION; SENSITIVITY ANALYSIS; NETWORK SYSTEMS;

1. Introduction.

The assumption of statistical independence is usual in reliability theory as it is in other parts of statistics and probability. The difficulties arising when the dependence cannot be ignored, are related both to the lack of information concerning the dependence structure as well as the possible increase in computational complexity. Recently, however, Hagstrom and Mak (1986) have shown that computing system reliability in the presence of correlated failures is not significantly harder than writing down the joint probability distribution and computing the system reliability when the components fail independently. Hence, it seems that developing satisfactory dependence models is the most important task.

In reliability theory dependence is frequently modelled in terms of associated random variables. (See Barlow and Proschan (1981).) Using this it is possible to develop bounds on the system reliabilty valid for a fairly large class of joint distributions for the component states. However, since these bounds are, in the worst cases, very crued, it is often preferred to work out the socalled excact expressions under the independence assumption, neglecting that this may be terribly unrealistic.

In the case of association it is known that this leads to underestimation of the system reliability if the system is series, whereas the converse holds for parallel systems. For more general types of systems no such results exists.

In the present report we consider two parametric dependence models. We shall refer to these models as the shock model and the stand-by model, respectively. The shock model is identical to the one considered in Boyles and Samaniego (1984) and Huseby (1986). In a sense these models can be viewed as dual to each other. However, in both cases one gets variables being associated.

Under the above models it is possible to provide some further characterizations of the occurence of over- and underestimation. These characterizations depend not only on the structure of the system under consideration, but also on certain parameters of the joint distribution. The report generalizes parts of Egeland (1985).

2. Basic notation and results.

In this section we review some basic results of reliability theory needed in this study. We start by introducing the notion of binary monotone system.

A *binary monotone system* (BMS) is an ordered pair (E,ϕ), where $E = \{1,...,n\}$ is a non-empty set of components, and $\phi=\phi(\mathbf{X})$ is a binary (0-1) non-decreasing function of the component state vector $\mathbf{X}$. The function ϕ is called the *structure function* of the system, and describes the state of the system; i.e. $\phi=1$ if the system is functioning and $\phi=0$ if the system is failed. Similarly, the i-th entry of the component state vector, X_i, is respectively 1 or 0 if the i-th component is functioning or failed, $i=1,...,n$.

It is well-known that ϕ is a socalled multilinear function of $\mathbf{X}$. That is, for some suitable function δ, ϕ may be expressed as:

$$(2.1.) \quad \phi(\mathbf{X}) = \sum_{A\subseteq E} \delta(A) \prod_{i\in A} X_i$$

The function δ is called the *signed domination function* of the system. The concept of domination was introduced in Satyanarayana and Prabhakar (1978). Huseby (1984) and Huseby (1985) provides a further study of this. Other recent papers on this subject are Barlow and Iyer (1985) and Hagstrom (1986).

We say that a BMS, (E,ϕ), is *trivial* if ϕ is constant w.r.t. $\mathbf{X}$. Otherwise, it is called *non-trivial.*

If A is a subset of E, then $\mathbf{X}^A$ denotes the subvector of $\mathbf{X}$ corresponding to the set A. If A_1, A_2,... are disjoint subsets of E, we shall use the notation $(\mathbf{x}^{A_1},\mathbf{x}^{A_2},...,\mathbf{x})$ denoting the vector $\mathbf{X}$ where the subvectors corresponding to the sets A_1, A_2,... have the values $\mathbf{x}^{A_1}$, $\mathbf{x}^{A_2}$,... respectively. The rest of the vector (i.e. entries corresponding to the set $E\setminus(A_1\cup A_2\cup\cdots)$) has the value $\mathbf{x}$. Thus f.ex. $\mathbf{X} = (\mathbf{1}^{A_1},\mathbf{0}^{A_2},\mathbf{x})$ means that the entries of $\mathbf{X}$ corresponding to the sets A_1 and A_2 have the values 1 and 0 respectively, while the rest of the vector is just specified to be $\mathbf{x}$.

By conditioning on some of the component state variables, say those corresponding to the set $A \subset E$, one obtains a socalled *minor system* of the original system. F.ex. given that $\mathbf{X}^A = (\mathbf{1}^{A_1},\mathbf{0}^{A_2})$ where $A_1 \cup A_2 = A$ and $A_1 \cap A_2 = \emptyset$, one obtains a minor system denoted by $(E \setminus A, \phi_{+A_1-A_2})$ (or alternatively $(E \setminus A, \phi_{-A_2+A_1})$) where the structure function $\phi_{+A_1-A_2}$ is defined on $\mathbf{X}^{E \setminus A}$ and given by:

$$(2.2.) \qquad \phi_{+A_1-A_2}(\cdot) = \phi(\mathbf{1}^{A_1}, \mathbf{0}^{A_2}, \cdot)$$

If either A_1 or A_2 is empty, we simply write ϕ_{-A_2} or ϕ_{+A_1} respectively.

A set $A \subseteq E$ is a *cut set* if $\phi_{-A}(\mathbf{X}^{E \setminus A}) = 0$ for all $\mathbf{X}^{E \setminus A}$. Similarly, a set $A \subseteq E$ is a *path set* if $\phi_{+A}(\mathbf{X}^{E \setminus A}) = 1$ for all $\mathbf{X}^{E \setminus A}$.

Of special relevance to this paper are the number of cut sets and the number of path sets of cardinality 1. For a BMS (E,ϕ) these are denoted by $c(\phi)$ and $s(\phi)$ respectively. If the system is trivial, we define $c(\phi) = s(\phi) = 0$. The following are some easily established properties of $c(\phi)$ and $s(\phi)$.

(2.3.) $\quad c(\phi) = 0 \Rightarrow c(\phi_{+A}) = 0 \;, \quad s(\phi) = 0 \Rightarrow s(\phi_{-A}) = 0.$

(2.4.) $\quad$ If $(E \setminus A, \phi_{-A})$ is non-trivial, then $c(\phi) > 0 \Rightarrow c(\phi_{-A}) > 0$.
If $(E \setminus A, \phi_{+A})$ is non-trivial, then $s(\phi) > 0 \Rightarrow s(\phi_{+A}) > 0$.

The reliability function of a BMS (E,ϕ) is denoted by h and is defined by:

$$(2.5.) \qquad h = h(\mathbf{p}) = \sum_{A \subseteq E} \delta(A) \prod_{i \in A} p_i$$

where δ is the signed domination function of the system and $\mathbf{p} = (p_1,\ldots\ldots,p_n)$ is the vector of component reliabilities, (i.e. $p_i = P(X_i=1)$, $i=1,\ldots,n$). When $p_1 = \cdots = p_n = p$, we simply write $h = h(p)$. (Observe that $h(p)$ is a polynomial in p of degree n or less.) If (E,ϕ) is non-trivial, then $h(\mathbf{0})=0$ and $h(\mathbf{1})=1$. Furthermore, if the component state variables are independent, then $h = \Pr(\phi=1)$ = The reliability of the system.

We close this section by listing some well-known, useful properties of the reliability function:

(2.6.) $h'(p)\big|_{p=0} = s(\phi)$, $h'(p)\big|_{p=1} = c(\phi)$.

(2.7.) $s(\phi) > 0 \quad \Rightarrow \quad h(p) \geq p$ for all $p \in [0,1]$.

(2.8.) $c(\phi) > 0 \quad \Rightarrow \quad h(p) \leq p$ for all $p \in [0,1]$.

If $\mathbf{p} = (p_1,....,p_n)$ and $\mathbf{q} = (q_1,....,q_n)$, then $(\mathbf{p}\ \mathbf{q})$ and $(\mathbf{p} \amalg \mathbf{q})$ denote the vectors $(p_1q_1,....,p_nq_n)$ and $(p_1 \amalg q_1,....,p_n \amalg q_n)$ respectively. ($\amalg$ denotes the socalled "ip-operator" and is defined by $(z_1 \amalg \cdots \amalg z_n) = 1 - (1-z_1) \cdots (1-z_n)$.) If $\mathbf{p}$ and $\mathbf{q}$ are vectors of probabilities, then:

(2.9.) $h(\mathbf{p}\ \mathbf{q}) \leq h(\mathbf{p})\ h(\mathbf{q})$

(2.10.) $h(\mathbf{p} \amalg \mathbf{q}) \geq h(\mathbf{p}) \amalg h(\mathbf{q})$

For more details and proofs we refer to Barlow and Proschan (1981).

3. The dependence models.

We start by presenting the shock model. This model was suggested by Boyles and Samaniego (1984) and can be viewed as a discrete analog to the well-known multivariate exponential distribution introduced by Marshall and Olkin (1967). It appears to be particulary useful in order to model socalled common cause failures. However, as shown in Huseby (1986), it may also serve as a tool for deriving fast algorithms for reliability calculations.

The model is based on the assumption that the failures of the components are caused by different types of "shocks" striking single components or groups of components. More precisely, we assume that for each non-empty subset A of the component set E, there exists a possible shock which, if it occurs, kills all the components in the set A and these *alone*. In order to describe the "shock status", for each non-empty $A \subseteq E$, we introduce a random variable Y_A being 1 if the shock striking the set A has not occurred and 0 otherwise.

The component state variables, i.e. the X_i-s, may now be expressed in terms of the Y_A-s as follows:

(3.1.) $$X_i = \prod_{A:\, i \in A} Y_A \quad , \quad i = 1,....,n.$$

We assume that the Y_A-s are independent and that $P(Y_A=1) = \theta_A$, $\emptyset \subset A \subseteq E$. Hence, the component reliabilities are given by:

$$(3.2.)\qquad p_i = \prod_{A:\, i\in A} \theta_A \quad , \qquad i = 1,\ldots,n.$$

The shocks striking single components, will be called *individual* shocks, while the others will be called *external* shocks.

If some of the θ_A-s are equal to 1, then clearly the corresponding Y_A-s may be left out in (3.1.). Thus, restricted models, i.e. models where only some of the shocks are present, may be derived as special cases of the general model. Especially, if θ_A=1 for all A with cardinality greater than 1, i.e. only individual shocks are present, then the X_i-s are independent.

We now turn to the stand-by model. This model is based on the assumption that the functioning of the components are ensured by different types of "stand-by components". Specifically, we assume that for each non-empty subset A of the component set E, there exists a stand-by component which if it functions, ensures that the set A, and this *alone,* functions. As for the shock model, we introduce status variables Z_A being 1 if the stand-by corresponding to the set A is functioning and 0 otherwise.

The X_i-s is expressed in terms of the Z_A-s as follows:

$$(3.3.)\qquad X_i = \coprod_{A:\, i\in A} Z_A \quad , \qquad i = 1,\ldots,n.$$

Assuming that the Z_A-s are independent and that $P(Z_A=1) = \mu_A$, $\emptyset\subset A\subseteq E$, implies that the component reliabilities in this case are given by:

$$(3.4.)\qquad p_i = \coprod_{A:\, i\in A} \mu_A \quad , \qquad i = 1,\ldots,n.$$

We say that a stand-by is *individual* if it affects a single component only. Otherwise, it is called *external.*

If some of the μ_A-s are equal to 0, then the corresponding Z_A-s may be left out in (3.3.). Thus again, restricted models, i.e. models where only some of the stand-by components are present, may be derived as special cases of the general model. Especially, if μ_A=0 for all A with cardinality greater than 1, i.e. only individual stand-by components are present, then the X_i-s are independent.

We observe that in both models the X_i-s are represented as increasing functions of independent variables. Hence, by standard results on associated random variables (see Barlow and Proschan (1981)), it follows that the X_i-s are associated.

4. Main results.

We shall now develop the main results of this report. Since the proofs typically are similar for the two models, we shall spend most of the time on the results concerning the shock model. The corresponding results for the stand-by model will be treated more briefly.

In the study of the shock model we have focused on cases where the external shock probabilities are low. Although many of the results either have obvious counterparts concerning external shocks with high probability, or directly apply to these cases, we consider such results far less important. Our main goal is to perform a sensitivity study. Thus, we concentrate on cases where the component state variables are not too strongly dependent. If the external shock probabilities are high, we have a more extreme type of dependence, and thus a sensitivity study is of limited value. In such cases the dependence has to be modelled explicitly, f.ex. by application of some plausible parametric model. Besides, it is a fact that in most practical applications the components are highly reliable, i.e. all the shock probabilities are low.

Of similar reasons in the study of the stand-by model we focus on cases where the external stand-by components have high failure probabilities.

Assume that (E,ϕ) is a BMS with reliability function h, and let $E=\{1,....,n\}$. As a start, we consider the simplest case where all the shocks except *one* are internal. Let $B\subseteq E$ be the set corresponding to the external shock. We assume that $\theta_A = \theta$ if $A=B$, and that $\theta_A=\pi_i$ if $A=\{i\}$, $i\in E$. Hence, the vector of component reliabilities is given by:

$$(4.1.) \qquad \mathbf{p} \;=\; (\theta\, \boldsymbol{\pi}^B, \boldsymbol{\pi})$$

If we ignore the dependence between the components (caused by the external shock), we would assess the system reliability to be:

$$(4.2.) \qquad R_1 \;=\; h(\theta\, \boldsymbol{\pi}^B, \boldsymbol{\pi})$$

The correct assessment is easily obtained by conditioning on the status variable of the external shock, and is given by:

$$(4.3.) \qquad R_2 \;=\; \theta\, h(\boldsymbol{\pi}) \;+\; (1-\theta)\, h(0^G, \boldsymbol{\pi})$$

where $G=E\backslash F$, and F is the union of minimal path sets which are not affected by the external shock.

Our main concern now is to study the sign of the difference $e(\theta,\boldsymbol{\pi}) = R_1 - R_2$ as θ and $\boldsymbol{\pi}$ vary. We denote this difference by $e(\theta,\boldsymbol{\pi})$. Clearly, the reliability is overestimated if $e(\theta,\boldsymbol{\pi})>0$, and underestimated if $e(\theta,\boldsymbol{\pi})<0$.

The first and easiest result concerns the case where B is a cut set. Note that in this case $h(\mathbf{0}^G,\boldsymbol{\pi})=0$. Hence, R_2 simply equals $\theta h(\boldsymbol{\pi})$.

Theorem 4.1. Consider the BMS (E,ϕ) described above. Assume especially that B, the set corresponding to the external shock, is a cut set.

If $c(\phi_{+(E\setminus B)}) > 0$, then for all $\theta, \pi_1,\dots,\pi_n \in [0,1]$:

$$(4.4.)\qquad h(\theta\, \boldsymbol{\pi}^B,\boldsymbol{\pi}) \leq \theta\, h(\boldsymbol{\pi}) \quad , \quad \text{i.e. } e(\theta,\boldsymbol{\pi}) \leq 0.$$

If $c(\phi_{+(E\setminus B)}) = 0$, then there exists an $\varepsilon>0$ $(\varepsilon\leq 1)$ such that for all $\theta, \pi_1,\dots,\pi_n \in [1-\varepsilon,1]$ we have:

$$(4.5.)\qquad h(\theta\, \boldsymbol{\pi}^B,\boldsymbol{\pi}) \geq \theta\, h(\boldsymbol{\pi}) \quad , \quad \text{i.e. } e(\theta,\boldsymbol{\pi}) \geq 0.$$

Proof: Assume first that $c(\phi_{+(E\setminus B)}) > 0$. Then by (2.9.) we get:

$$(4.6.)\qquad h(\theta\, \boldsymbol{\pi}^B,\boldsymbol{\pi}) \leq h(\theta\mathbf{1}^B,\mathbf{1})\, h(\boldsymbol{\pi}) \quad , \quad \text{for all } \theta, \pi_1,\dots,\pi_n \in [0,1].$$

Now, clearly $h(\theta\mathbf{1}^B,\mathbf{1})$ is the reliability function of $\phi_{+(E\setminus B)}$, evaluated at $(\theta\mathbf{1}^B)$. Hence, by (2.8.) and the assumption it follows that:

$$(4.7.)\qquad h(\theta\mathbf{1}^B,\mathbf{1}) \leq \theta \quad , \quad \text{for all } \theta\in[0,1].$$

By combining (4.6.) and (4.7.) the inequality (4.4.) follows.

Assume then that $c(\phi_{+(E\setminus B)}) = 0$, and consider the error function e when $\boldsymbol{\pi} = \mathbf{1}$. We may assume that (E,ϕ) is non-trivial since (4.5.) is obvious in the trivial case. Hence, especially $h(\mathbf{1}) = 1$ and we get:

$$(4.8.)\qquad e(\theta,\mathbf{1}) = h(\theta\mathbf{1}^B,\mathbf{1}) - \theta$$

As already mentioned, the first term of (4.8.) is the reliability function of $\phi_{+(E\setminus B)}$, evaluated at $(\theta\mathbf{1}^B)$. Hence, by (2.6.) we get that:

(4.9.) $\frac{\partial}{\partial\theta} e(\theta, \mathbf{1})\Big|_{\theta=1} = c(\phi_{+(E\setminus B)}) - 1 = -1.$

Since, $(\partial/\partial\theta)\, e(\theta,\boldsymbol{\pi})$ obviously is a continous function w.r.t. θ and $\boldsymbol{\pi}$ (being a polynomial in θ and $\boldsymbol{\pi}$), it follows that for some $\varepsilon>0$ ($\varepsilon\leq 1$) we have for all θ, $\pi_1,...,\pi_n \in [1-\varepsilon,1]$ that:

(4.10.) $\frac{\partial}{\partial\theta} e(\theta,\boldsymbol{\pi}) \geq -\delta.$

where $\delta>0$ is a suitable number.

By Taylor's formula and the obvious fact that $e(1,\boldsymbol{\pi})=0$, we then get that for all θ, $\pi_1,...,\pi_n \in [1-\varepsilon,1]$ we have:

(4.11.) $e(\theta,\boldsymbol{\pi}) = e(1,\boldsymbol{\pi}) + \frac{\partial}{\partial\theta} e(\theta_0,\boldsymbol{\pi})\,(\theta - 1) \geq 0 + \delta(1 - \theta) > 0$

where $\theta_0 \in [\theta,1]$ is suitably chosen. Thus, (4.5.) is proved. □

Using Theorem 4.1. as a tool we can now extend the result to the case where the external shock is not fatal to the system, i.e. B is not a cut set.

Theorem 4.2. Consider the BMS (E,ϕ) described above. Assume now that B, the set corresponding to the external shock, is not a cut set. Moreover, let F be the union of the minimal path sets which are not affected by the shock, i.e. those minimal path sets P such that $P\cap B=\emptyset$ and let $G=(E\setminus F)$. Finally, let $\mathbb{C}$ be the family of sets $C\subseteq F$ such that $(B,[\phi_{+(G\setminus B)}]_{+C-(F\setminus C)})$ is non-trivial.

If $c([\phi_{+(G\setminus B)}]_{+C-(F\setminus C)}) > 0$ for all $C\in\mathbb{C}$ then for all θ, $\pi_1,....,\pi_n \in [0,1]$ we have:

(4.12.) $h(\theta\boldsymbol{\pi}^B,\boldsymbol{\pi}) \leq \theta\, h(\boldsymbol{\pi}) + (1-\theta)\, h(\mathbf{0}^G,\boldsymbol{\pi})$, i.e. $e(\theta,\boldsymbol{\pi}) \leq 0.$

If $c([\phi_{+(G\setminus B)}]_{+C-(F\setminus C)}) = 0$ for all $C\in\mathbb{C}$ then there exists an $\varepsilon>0$ ($\varepsilon\leq 1$) such that for all θ, $\pi_1,....,\pi_n \in [1-\varepsilon,1]$ we have:

(4.13.) $h(\theta\boldsymbol{\pi}^B,\boldsymbol{\pi}) \geq \theta\, h(\boldsymbol{\pi}) + (1-\theta)\, h(\mathbf{0}^G,\boldsymbol{\pi})$, i.e. $e(\theta,\boldsymbol{\pi}) \geq 0.$

Proof: We introduce the notation:

(4.14.) $P(C) = [\prod_{i\in C} \pi_i]\,[\prod_{i\in F\setminus C} (1 - \pi_i)]$

Assume first that $c([\phi_{+(G\setminus B)}]_{+C-(F\setminus C)}) > 0$ for all $C\in \mathcal{C}$. By conditioning on the state variables of the components in F, we get:

$$(4.15.)\quad h(\theta\boldsymbol{\pi}^B,\boldsymbol{\pi}) = \sum_{C\subseteq F} h(\mathbf{1}^C,\mathbf{0}^{F\setminus C},\theta\boldsymbol{\pi}^B,\boldsymbol{\pi})\ P(C)$$

$$= \sum_{C\in \mathcal{C}} h(\mathbf{1}^C,\mathbf{0}^{F\setminus C},\theta\boldsymbol{\pi}^B,\boldsymbol{\pi})\ P(C) + \sum_{C\notin \mathcal{C}} h(\mathbf{1}^C,\mathbf{0}^{F\setminus C},\boldsymbol{\pi})\ P(C)$$

[by observing that $h(\mathbf{1}^C,\mathbf{0}^{F\setminus C},\theta\boldsymbol{\pi}^B,\boldsymbol{\pi}) = h(\mathbf{1}^C,\mathbf{0}^{F\setminus C},\boldsymbol{\pi})$ for all $C\in \mathcal{C}$ since these sets correspond to trivial systems, i.e. the reliabilities of the components in G does not affect the system reliability.]

$$\leq \theta \sum_{C\in \mathcal{C}} h(\mathbf{1}^C,\mathbf{0}^{F\setminus C},\boldsymbol{\pi})\ P(C) + \sum_{C\notin \mathcal{C}} h(\mathbf{1}^C,\mathbf{0}^{F\setminus C},\boldsymbol{\pi})\ P(C)$$

[by Theorem 4.1. and the assumption since the external shock obviously is fatal to the system $(G, [\phi_{+(G\setminus B)}]_{+C-(F\setminus C)})$ for all $C\in \mathcal{C}$.]

$$= \theta \sum_{C\subseteq F} h(\mathbf{1}^C,\mathbf{0}^{F\setminus C},\boldsymbol{\pi})\ P(C) + (1-\theta) \sum_{C\notin \mathcal{C}} h(\mathbf{1}^C,\mathbf{0}^{F\setminus C},\boldsymbol{\pi})\ P(C)$$

$$= \theta \sum_{C\subseteq F} h(\mathbf{1}^C,\mathbf{0}^{F\setminus C},\boldsymbol{\pi})\ P(C) + (1-\theta) \sum_{C\subseteq F} h(\mathbf{1}^C,\mathbf{0}^{F\setminus C},\mathbf{0})\ P(C)$$

[by observing that $h(\mathbf{1}^C,\mathbf{0}^{F\setminus C},\boldsymbol{\pi}) = h(\mathbf{1}^C,\mathbf{0}^{F\setminus C},\mathbf{0})$ for all $C\notin \mathcal{C}$, and that $h(\mathbf{1}^C,\mathbf{0}^{F\setminus C},\mathbf{0}) = 0$ for $C\in \mathcal{C}$.]

$$= \theta\, h(\boldsymbol{\pi}) + (1-\theta)\, h(\mathbf{0}^G,\boldsymbol{\pi})\,.$$

Hence (4.12.) is proved.

The proof of (4.13.) follows by noting that by Theorem 4.1. the inequality in (4.15.) is reversed when $c([\phi_{+(G\setminus B)}]_{+C-(F\setminus C)})=0$ for all $C\in \mathcal{C}$. □

Although the above theorems are developed in the context of shock models, they may be viewed more generally as results on reliability functions. By applying this general interpretation, the theorems may be used to handle cases where more than one external shock is present. We illustrate this by an example.

Example 4.3. Consider the network system (E,ϕ) illustrated in Figure 4.1. which functions if node s and node t can communicate through the network. The components of the system are denoted by 1, 2, ..., 6, and the corresponding internal shock probabilities are $(1-\pi_1)$, $(1-\pi_2)$, ..., $(1-\pi_6)$, respectively. We assume that the system is affected by two external shocks with probabilities $(1-\theta_1)$ and $(1-\theta_2)$, and that the corresponding sets are $B_1=\{1,2,3\}$ and $B_2=\{3,4,5,6\}$ respectively.

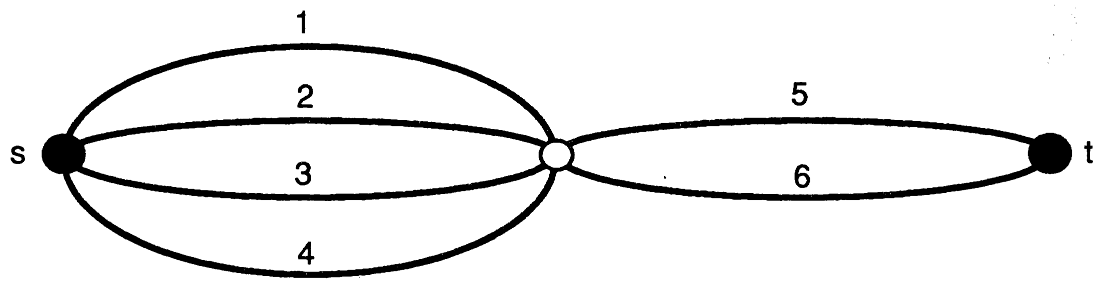


Figure 4.1. Graph of a network system.

Now, let $\mathbf{p}_1=(p_{11},.....,p_{16})$ and $\mathbf{p}_2=(p_{21},....,p_{26})$ be given by:

(4.16.) $\quad \mathbf{p}_1 = (\theta_1 \boldsymbol{\pi}^{B_1}, \boldsymbol{\pi})$ and $\mathbf{p}_2 = (\theta_2 \mathbf{p}_1^{B_2}, \mathbf{p}_1)$.

It is easy to see that ignoring the dependence caused by the two external shocks implies that the system reliability is assessed to be:

(4.17.) $\quad R_1 = h(\mathbf{p}_2)$.

The correct value is obtained by conditioning on the status variables of the two external shocks, and observing that the shock corresponding to the set B_2 is fatal to the system, and is given by:

(4.18.) $\quad R_2 = \theta_2[\ \theta_1\ h(\boldsymbol{\pi}) + (1-\theta_1)\ h(\mathbf{0}^{B_1}, \boldsymbol{\pi})\ ]$

Since B_2 is a cut set, and $c(\phi_{+(E\backslash B_2)})=0$ we may apply Theorem 4.1. to $h(\mathbf{p}_2)$ and get:

(4.19.) $\quad h(\mathbf{p}_2) \geq \theta_2\ h(\mathbf{p}_1)$

provided that $\theta_2, p_{11},...,p_{16} \in [1-\varepsilon_1, 1]$ for some suitable $\varepsilon_1>0$ $(\varepsilon_1 \leq 1)$. Furthermore, B_1 is not a cut set. Hence, we can apply Theorem 4.2. to $h(\mathbf{p}_1)$. Let F denote the union of minimal path sets, P, such that $P \cap B_1 = \emptyset$, i.e. $F=\{4,5,6\}$, let $G=(E \setminus F)$, i.e. $G=B_1$, and let $\mathcal{C}$ denote the family of sets $C \subseteq F$ such that $(B_1, [\phi_{+(G \setminus B_1)}]_{+C-(F \setminus C)})$ is non-trivial. It is easy to see that the only set in $\mathcal{C}$ is $C=\{5,6\}$. Hence, since $c([\phi_{+(G \setminus B_1)}]_{+C-(F \setminus C)})=0$, we get:

$$\text{(4.20.)} \qquad h(\mathbf{p}_1) \geq \theta_1\, h(\boldsymbol{\pi}) + (1-\theta_1)\, h(\mathbf{0}^{B_1}, \boldsymbol{\pi})$$

provided that $\theta_1, \pi_1, \ldots, \pi_6 \in [1-\varepsilon_2, 1]$ for some suitable $\varepsilon_2 > 0$ $(\varepsilon_2 \leq 1)$.

By combining (4.19.) and (4.20.) we get that $R_1 \geq R_2$ whenever $\theta_1, \theta_2, \pi_1, \ldots, \pi_6 \in [1-\varepsilon, 1]$ where $\varepsilon > 0$ $(\varepsilon \leq 1)$ is some suitable number. That is, in this case the reliability is overestimated if the dependence is ignored.

In the above calculations we treated the external shocks in two steps. The first step concerned the shock striking B_2, while the second concerned the one striking B_1. It is important to notice that these steps cannot be interchanged. If we start out by treating the shock striking B_1, then it turns out to be impossible to obtain any useful inequalities in the second step. In order to take a closer look at this, we introduce $\mathbf{q}_1 = (q_{11}, \ldots, q_{16})$ and $\mathbf{q}_2 = (q_{21}, \ldots, q_{26})$ given by:

$$\text{(4.21.)} \qquad \mathbf{q}_1 = (\theta_2 \boldsymbol{\pi}^{B_2}, \boldsymbol{\pi}) \quad \text{and} \quad \mathbf{q}_2 = (\theta_1 \mathbf{q}_1^{B_1}, \mathbf{q}_1)\,.$$

We obviously have that $\mathbf{p}_2 = \mathbf{q}_2$. Hence, especially $R_1 = h(\mathbf{q}_2)$.

Since B_1 is not a cut set, we may apply Theorem 4.2. to $h(\mathbf{q}_2)$ and get:

$$\text{(4.22.)} \qquad h(\mathbf{q}_2) \geq \theta_1\, h(\mathbf{q}_1) + (1-\theta_1)\, h(\mathbf{0}^{B_1}, \mathbf{q}_1)\,.$$

provided that $\theta_1, q_{11}, \ldots, q_{16} \in [1-\varepsilon_1, 1]$ for some suitable $\varepsilon_1 > 0$ $(\varepsilon_1 \leq 1)$.

In order to proceed we may now try to apply Theorem 4.1. to $h(\mathbf{q}_1)$ and $h(\mathbf{0}^{B_1}, \mathbf{q}_1)$. It is easily established that $h(\mathbf{q}_1) \geq \theta_2\, h(\boldsymbol{\pi})$ for $\theta, \pi_1, \ldots, \pi_6$ sufficiently close to 1. However, $h(\mathbf{0}^{B_1}, \mathbf{q}_1)$ is equal to the reliability function of $(E \setminus B_1, \phi_{-B_1})$, evaluated at $\mathbf{q}_1^{E \setminus B_1}$, [See (2.2.)] and $c(\phi_{-B_1})=1$. Hence, we get that $h(\mathbf{0}^{B_1}, \mathbf{q}_1) \leq \theta_2\, h(\mathbf{0}^{B_1}, \boldsymbol{\pi})$. Thus, no conclusion is obtainable in this case. □

As shown in the above example, it is possible to extend the results given in Theorem 4.1. and Theorem 4.2. considerably. However, especially if many of the external shocks are non-fatal to the system, the conditions for under- and overestimation soon become very involved. In some cases, nice characterizations can be obtained. In the following two theorems we consider a BMS (E, ϕ) where $\mathcal{A}$ is the family of sets corresponding to the external shocks. In particular, we assume that for each set, $A\in\mathcal{A}$, the corresponding external shock occurs with probability $(1-\theta_A)$. Let $\boldsymbol{\theta}$ be the vector of θ_A-s, (in some arbitrary order). As before the internal shocks occur with probabilities $\pi_1,\ldots\ldots,\pi_n$, respectively. Finally, we define the error function:

(4.23.) $\quad e(\boldsymbol{\theta},\boldsymbol{\pi})$ = (The reliability assessed assuming independence) - (Correct value.)

Theorem 4.4. Consider the BMS (E,ϕ) described above. Assume especially that each $A\in\mathcal{A}$ is fatal to the system.

If $c(\phi_{+(E\setminus A)}) > 0$ for each $A\in\mathcal{A}$, then for all θ_A $(A\in\mathcal{A})$, $\pi_1,\ldots\ldots,\pi_n\in[0,1]$, we have that $e(\boldsymbol{\theta},\boldsymbol{\pi}) \leq 0$, i.e. the reliability is underestimated.

If $c(\phi_{+(E\setminus A)}) = 0$ for each $A\in\mathcal{A}$, then there exists an $\varepsilon>0$ $(\varepsilon\leq 1)$ such that for all θ_A $(A\in\mathcal{A})$, $\pi_1,\ldots\ldots,\pi_n\in[1-\varepsilon,1]$ we have that $e(\boldsymbol{\theta},\boldsymbol{\pi}) \geq 0$, i.e. the reliability is overestimated.

Proof: The result follows directly by applying Theorem 4.1. as in Example 4.3.

Observe that if $c(\phi)=0$, then by (2.3.) $c(\phi_{+(E\setminus A)})=0$ for all $A\subseteq E$. Hence, the condition for overestimation is often very easy to verify.

Theorem 4.5. Consider the BMS (E,ϕ) described above. Assume especially that each $A\in\mathcal{A}$ is non-fatal to the system. Moreover, for each $A\in\mathcal{A}$, let F_A be the union of minimal path sets which are not affected by the shock, i.e. those minimal path sets P such that $P\cap A=\emptyset$ and let $G_A=(E\setminus F_A)$. Finally, let $\mathcal{C}_A$ be the family of sets $C\subseteq F_A$ such that $(A, [\phi_{+(G_A\setminus A)}]_{+C-(F_A\setminus C)})$ is non-trivial.

If $c(\,[\phi_{+(G_A\setminus A)}]_{+C-(F_A\setminus C)}\,) > 0$ for all $C\in\mathcal{C}_A$ and $A\in\mathcal{A}$, then for all θ_A $(A\in\mathcal{A})$, $\pi_1,\ldots,\pi_n\in[0,1]$, we have that $e(\boldsymbol{\theta},\boldsymbol{\pi}) \leq 0$, i.e. the reliability is underestimated.

Proof: The theorem follows by applying Theorem 4.2. as in Example 4.3. and noting that by (2.4.) $c(\,[\phi_{+(G_A\setminus A)}]_{+C-(F_A\setminus C)}\,) > 0$ for all $C\in\mathcal{C}_A$ implies that $c(\,[\,[\phi_{+(G_A\setminus A)}]_{+C-(F_A\setminus C)}\,]_{-B}\,) > 0$ for all $C\subseteq F_A$ and $B\subseteq A$, such that $(A\setminus B, [\,[\phi_{+(G_A\setminus A)}]_{+C-(F_A\setminus C)}\,]_{-B})$ is non-trivial. □

It is possible to formulate a sufficient condition for overestimation of the reliability in the case of shocks which are non-fatal to the system. However, as one may suspect, this is rather complicated and thus not very useful in practical situations. This topic is discussed further in Section 6.

We observe that in all the above theorems the sign of the error depends both on the "cut structure" of the system and the different shock probabilities. In a given practical situation it is (at least in principle) possible to verify the cut structure condition simply by examining the structure of the system. Information concerning the shock probabilities, however, may be far more difficult to get. Still it is possible to get around this problem if the sets corresponding to the external shocks as well as the marginal component reliabilities are known. This is done as follows:

By examining the sets corresponding to the external shocks and the cut structure of the system, one determines if the given system satisfies the cut structure condition of any of the above theorems. If the cut structure indicates that the reliability is underestimated, i.e. the relevant minors contain cut sets of cardinality one, then clearly the shock probabilities need not be considered. That is, the reliability is underestimated for all possible shock probabilities.

However, if the cut structure indicates overestimation, i.e. none of the relevant minors have cut sets of cardinality one, then we must examine the θ_A-s and π_i-s more carefully. More specifically, we must determine whether we have:

(4.24.) $\theta_A \in [1-\varepsilon,1]$ for all $A \in \mathcal{A}$, and $\pi_i \in [1-\varepsilon,1]$ for all $i \in E$.

where $\varepsilon>0$ ($\varepsilon \leq 1$) is determined from the structure.

Let p_i denote the marginal reliability of the i-th component, and $\mathcal{A}_i$ denote the family of sets $A \in \mathcal{A}$ such that $i \in A$, i=1,....,n. Then we have:

(4.25.) $$p_i = \pi_i (\prod_{A \in \mathcal{A}_i} \theta_A), \quad i=1,...,n.$$

Now, a sufficient condition for (4.24.) to be true may be expressed in terms of the p_i-s simply as:

(4.26.) $$p_i \in [1-\varepsilon,1] \quad , \quad i=1,....,n.$$

Thus, we see that it is not necessary to know the excact values of the shock probabilities in order to obtain a conclusion. This simple observation extends the usefulness of the above results considerably.

In the last theorem on the shock model we show that if there are many non-fatal shocks, then even the cut structure condition for overestimation given in Theorem 4.5. is rarely satisfied. Similarly, if there are many fatal shocks (i.e. fatal to the system), then the cut structure condition for underestimation given in Theorem 4.4. is rarely satisfied.

Theorem 4.6. Consider the BMS (E,ϕ) described above, and let $\mathcal{A}$ be the family of sets corresponding to the external shocks. Furthermore, let $\mathcal{P}$ be the family of path sets of cardinality greater than one which are not cut sets, and $\mathcal{Q}$ be the family of minimal cut sets. For simplicity we assume that every component in E is contained in at least one minimal cut set.

If $\mathcal{Q}\subseteq\mathcal{A}$, then the cut structure condition for underestimation given in Theorem 4.4. is satisfied if and only if the system is a series system of all the components, that is $\phi(\mathbf{X}) = X_1 X_2 \cdots X_n$.

If $\mathcal{P}\subseteq\mathcal{A}$, and $\mathcal{P}$ is non-empty, then the cut structure condition for overestimation given in Theorem 4.5. is satisfied if and only if the system is a parallel system of all the components, that is $\phi(\mathbf{X}) = X_1 \amalg X_2 \amalg \cdots \amalg X_n$.

Proof: It is very easy to see that the "if"-parts of the theorem are true. [Indeed since the variables in our models always are associated, it follows by standard results (see Barlow and Proschan (1981)) that the reliability is underestimated if the system is series and overestimated if the system is parallel. Thus, Theorem 4.4. and Theorem 4.5. are not needed to characterize the error function in this case.]

Assume first that $\mathcal{Q}\subseteq\mathcal{A}$, and that the cut structure condition for underestimation given in Theorem 4.4. is satisfied. Hence, especially $c(\phi_{+(E\setminus A)}) > 0$ for each $A\in\mathcal{Q}$. However, if $A\in\mathcal{Q}$, then $(A,\phi_{+(E\setminus A)})$ is a parallel system. Thus, $c(\phi_{+(E\setminus A)}) > 0$ implies that the set A has cardinality one. Since this is true for all $A\in\mathcal{Q}$, and since we have assumed that every component is contained in at least one minimal cut set, it follows that (E,ϕ) is a series system of all the components, i.e. $\phi(\mathbf{X}) = X_1 X_2 \cdots X_n$.

Assume then that $\mathcal{P}\subseteq\mathcal{A}$, $\mathcal{P}$ is non-empty, and that the cut structure condition for overestimation given in Theorem 4.5. is satisfied. Now, for each $A\in\mathcal{P}$ let G_A and F_A be defined as in Theorem 4.5. If $A\in\mathcal{P}$, then by definition A is a path set and F_A contains the union of all minimal path sets, P, such that $P\cap A=\emptyset$. Hence, $(G_A\setminus A)$ cannot be a path set, and thus $(A, [\phi_{+(G_A\setminus A)}]_{-F_A})$ is non-trivial. Especially, it follows by the assumption that $c([\phi_{+(G_A\setminus A)}]_{-F_A}) = 0$, implying that $(A, [\phi_{+(G_A\setminus A)}]_{-F_A})$ cannot be a series system. Thus, A cannot be a minimal path set. Since this is true for all $A\in\mathcal{P}$, it follows that all the minimal path sets of the system either are of cardinality one or are cut sets. Assume now that there exists a minimal path set P_0 of cardinality greater than one, i.e. P_0 is a cut set. Then it is easy to see that $P\cap P_0\neq\emptyset$ for all minimal path sets P (See Huseby (1984) p.11). Hence, since P_0 is minimal, this implies that every minimal path set has cardinality greater than one, and thus is a cut set. However, this implies that every path set (minimal or not) is also a cut set, contradicting that the family $\mathcal{P}$ is non-empty. Hence, we conclude that every minimal path set is of cardinality one. Since we have assumed that every component is contained in at least one minimal cut set, we get that (E,ϕ) is a parallel system of all the components, i.e. $\phi(X) = X_1 \amalg X_2 \amalg \cdots \amalg X_n$. □

We close this section by briefly presenting the corresponding results on the stand-by model. Since the proofs are completely analogous, we skip them here. Note, however, the duality between the results on the two models. While the conditions in the theorems on the shock model were formulated in terms of the cut structure, the corresponding results on the stand-by model are formulated in terms of the path structure.

We now consider a BMS (E,ϕ) where $\mathcal{A}$ is the family of sets corresponding to external stand-by components. In particular, we assume that for each set $A \in \mathcal{A}$, the corresponding external stand-by is functioning with probability μ_A. Let $\boldsymbol{\mu}$ be the vector of the μ_A-s (in some arbitrary order), and let $\boldsymbol{\pi} = (\pi_1,\ldots,\pi_n)$ be the vector of internal stand-by reliabilities.

If $\mathcal{A}$ contains only one set, the reliability assessed when assuming independence is given by:

$$(4.27.) \quad R_1 = h(\mu \amalg \boldsymbol{\pi}^A, \boldsymbol{\pi}) .$$

where μ is the reliability of the external stand-by. The correct value is:

$$(4.28.) \quad R_2 = \mu\, h(\mathbf{1}^G, \boldsymbol{\pi}) + (1-\mu)\, h(\boldsymbol{\pi}) .$$

where $G=E\backslash F$, and F is the union of minimal cut sets C such that $C \cap A = \emptyset$. By using (2.6.), (2.7.), (2.10.) and Taylor's formula, a result analogous to Theorem 4.1. may be developed. More generally, we define the error function:

(4.29.) $e(\boldsymbol{\mu},\boldsymbol{\pi})$ = (The reliability assessed assuming independence) - (Correct value).

Theorem 4.7. Consider the BMS (E,ϕ) described above. Assume especially that each $A \in \mathcal{A}$ is a path set of the system.

If $s(\phi_{-(E\backslash A)}) > 0$ for each $A \in \mathcal{A}$, then for all μ_A $(A \in \mathcal{A})$, $\pi_1,\ldots,\pi_n \in [0,1]$ we have that $e(\boldsymbol{\mu},\boldsymbol{\pi}) \geq 0$, i.e. the reliability is overestimated.

If $s(\phi_{-(E\backslash A)}) = 0$ for each $A \in \mathcal{A}$, then there exists an $\varepsilon > 0$ $(\varepsilon \leq 1)$ such that for all μ_A $(A \in \mathcal{A})$,

$\pi_1,\ldots,\pi_n \in [0,\varepsilon]$ we have that $e(\boldsymbol{\mu},\boldsymbol{\pi}) \leq 0$, i.e. the reliability is underestimated. □

Note that if $s(\phi)=0$, then by (2.3.) $s(\phi_{-(E\setminus A)}) = 0$ for each $A\subseteq E$. Hence, the condition for underestimation is often very easy to verify.

Theorem 4.8. Consider the BMS (E,ϕ) described above. Assume especially that $A\in \mathcal{A}$ is not a path set of the system. Moreover, for each $A\in \mathcal{A}$, let F_A be the union of minimal cut sets which are not affected by the stand-by, i.e. those minimal cut sets C such that $C\cap A=\emptyset$, and let $G_A=(E\setminus F_A)$. Finally, let $\mathcal{C}_A$ be the family of sets $C\subseteq F_A$ such that $(A, [\phi_{-(G_A\setminus A)}]_{+C-(F_A\setminus C)})$ is non-trivial.

If $s([\phi_{-(G_A\setminus A)}]_{+C-(F_A\setminus C)}) > 0$ for all $C\in \mathcal{C}_A$, then for all μ_A $(A\in \mathcal{A})$, $\pi_1,\ldots,\pi_n \in [0,1]$, we have that $e(\boldsymbol{\mu},\boldsymbol{\pi}) \geq 0$, i.e. the reliability is overestimated. □

Also in this case it is possible to formulate a sufficient condition for underestimation of the reliability in the case of stand-by components corresponding to non-path sets. However, as in the shock model case, the condition is too complicated to be of practical use.

In both the above theorems the sign of the error depends on the "path structure" of the system and the different stand-by reliabilities. However, by using an argument similar to the one we used in the case of the shock model, it can be seen that it is sufficient to know the marginal reliabilities of the components instead of the excact stand-by reliabilities.

Finally, we present the result on the stand-by model corresponding to Theorem 4.6.

Theorem 4.9. Consider the BMS (E,ϕ) described above, and let $\mathcal{A}$ be the family of sets corresponding to the external stand-by components. Furthermore, let $\mathcal{Q}$ be the family of cut sets of cardinality greater than one which are not path sets, and let $\mathcal{P}$ be the family of minimal path sets.

For simplicity we assume that every component in E is contained in at least one minimal path set.

If $\mathcal{P}\subseteq\mathcal{A}$, then the path structure condition for overestimation given in Theorem 4.7. is satisfied if and only if the system is a parallel system of all the components, i.e. $\phi(\mathbf{X}) = X_1 \amalg \cdots \amalg X_n$.

If $\mathcal{Q}\subseteq\mathcal{A}$ and $\mathcal{Q}$ is non-empty, then the path structure condition for underestimation given Theorem 4.8. is satisfied if and only if the system is a series system of all the components, i.e.

$\phi(X) = X_1 X_2 \cdots X_n$. □

5. Applications to network systems.

In this section we present some applications of our results to network systems. It is easily seen that the results given in this section may be extended to cover any type of network system. However, in order to limit the presentation we have chosen to investigate dependence models for socalled k-terminal undirected network systems. An example of such a system is shown in Figure 5.1. (k=3).

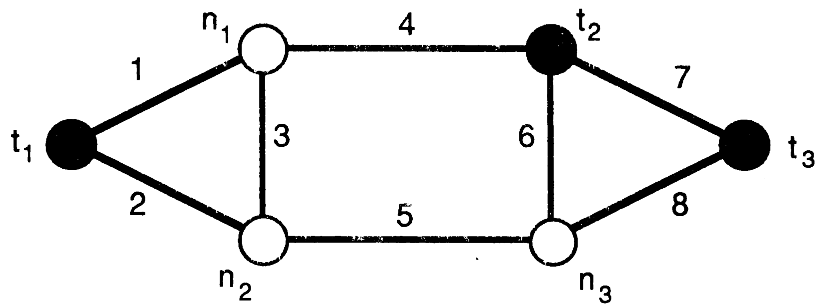


Figure 5.1. A 3-terminal undirected network system.

The system functions if and only if the three terminal nodes t_1, t_2 and t_3 can communicate through the network. We assume that the nodes of the system are perfect. Hence, the components of the system are the edges (denoted by the numbers 1,....,8).

Now, assume that (E,ϕ) is a k-terminal undirected network system with perfect nodes, i.e. $E = \{1,....,n\}$ is the set of edges of the network. Furthermore, let $V = \{v_1,.....,v_m\}$ denote the node set of the network, and $T \subseteq V$ the set of terminals. Thus, $|T|$ = (the cardinality of T) = k, and we of course have that $2 \leq k \leq m$. For each $v \in V$ we let E(v) denote the set of edges incident to the node v. Finally, we introduce two families of subsets of E given by:

(5.1.) $\quad \mathcal{N} = \{E(v) : v \in V\}$

(5.2.) $\quad \mathcal{M}$ = The family of minimal circuit sets of the network.

We now consider two types of dependence models for (E,ϕ), based on respectively the shock model and the stand-by model.

Model 1. Assume that the dependence between the components of (E,ϕ) is such that it is reasonable to use a shock model. It then remains to select the family $\mathcal{A}$ of sets corresponding to the external shocks. Intuitively, it seems natural to concentrate on shocks striking edges being "close" to each other in the network. Clearly, if $v \in V$, then all the components incident to this node will be close to each other. Motivated by this we let $\mathcal{A}$ be a subfamily of $\mathcal{N}$. Specifically, we choose a suitable set $S \subseteq V$ such that $|E(v)|>1$ for each $v \in S$, and define:

(5.3.) $\quad \mathcal{A} = \{E(v) : v \in S\}$

Without loss of generality we may assume that $S = \{v_1,\ldots,v_s\}$ where $s \leq m$ is the number of external shocks. Thus, letting $A_j=E(v_j)$, $j=1,\ldots,s$, we may write $\mathcal{A} = \{A_1,\ldots,A_s\}$. We assume that for each set $A_j \in \mathcal{A}$ the corresponding external shock occurs with probability $(1-\theta_j)$, and that for each component $i \in E$ the corresponding internal shock occurs with probability $(1-\pi_i)$. Finally, let $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_s)$ and $\boldsymbol{\pi} = (\pi_1,\ldots,\pi_n)$, and introduce the error function $e(\boldsymbol{\theta},\boldsymbol{\pi})$ = The error caused by neglecting the dependence. (See (4.23.)).

Note that using this model, we may interpret the external shocks as shocks striking the nodes in S. If a node is killed by a shock, then the components incident to this node, cannot communicate through this node and are thus eliminated from the system. According to this interpretation, the nodes in S may be treated as components having reliabilities $\theta_1,\ldots,\theta_s$, respectively, while the edges may be treated as having reliabilities $\pi_1,\ldots,\pi_n$, respectively. With this extended component set all the shocks may be viewed as internal shocks and thus the components (in the extended sense of the word) become independent.

On the other hand, neglecting the dependence caused by the external shocks, is equivalent to replacing θ_j by 1, $j=1,\ldots,s$ and π_i by p_i, $i=1,\ldots,n$, where $p_1,\ldots,p_n$ are given by:

(5.4.) $$p_i = \pi_i \Big[\prod_{j\,:\, i \in A_j} \theta_j\Big], \quad i = 1,\ldots,n.$$

Thus, if we use the interpretation of the external shocks as shocks striking the nodes, then neglecting dependence may be viewed as a transformation of systems with unreliable nodes into systems with perfect nodes.

Several efficient algorithms for excact reliability computations in the case of independent components apply to networks with unreliable nodes. (See f.ex. Satyanarayana (1982) and Wood (1985)). Still, in many cases it may be very useful to perform a transformation as indicated above. Especially, when the reliability of a network system is computed using socalled edge factoring (See Satyanarayana and Chang (1983)), unreliable nodes can cause problems. This is due to the fact that it may be impossible to obtain network representations of the system after a factoring in the case of unreliable nodes.

Obviously, the above transformation will not preserve the reliability of the system. However, as we shall see, by using results from Section 4. in some cases, it may be possible to determine the sign of the error. The most important result is provided in the following theorem:

Theorem 5.1. Consider the k-terminal undirected network system (E,ϕ) described above. Assume now that $S \cap T = \varnothing$ (i.e. the external shocks strike non-terminal nodes only), and that $|A_j| \leq 3$, $j=1,....,s$. Then $e(\boldsymbol{\theta},\boldsymbol{\pi}) \leq 0$ for all $\theta_1,....,\theta_s, \pi_1,....,\pi_n \in [0,1]$, i.e. the reliability is underestimated.

Proof: We start out by partitioning $\mathcal{A}$ into two families $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{A}_1$ contains the sets corresponding to system fatal shocks, and $\mathcal{A}_2=\mathcal{A}\backslash\mathcal{A}_1$. For each $A\in \mathcal{A}_2$ we define F_A, G_A and $\mathcal{C}_A$ as in Theorem 4.5.

By combining Theorem 4.1. and Theorem 4.2. as we did in Example 4.3. and using (2.4.) as we did in the proof of Theorem 4.5. it follows that $e(\boldsymbol{\theta},\boldsymbol{\pi}) \leq 0$ for all $\theta_1,....,\theta_s, \pi_1,....,\pi_n \in [0,1]$, if:

(5.5.) $\quad c(\phi_{+(E\backslash A)}) > 0 \quad , \quad$ for all $A\in \mathcal{A}_1$,

and

(5.6.) $\quad c(\, [\phi_{+(G_A\backslash A)}]_{+C-(F_A\backslash C)}) > 0 \quad , \quad$ for all $C\in \mathcal{C}_A$ and $A\in \mathcal{A}_2$.

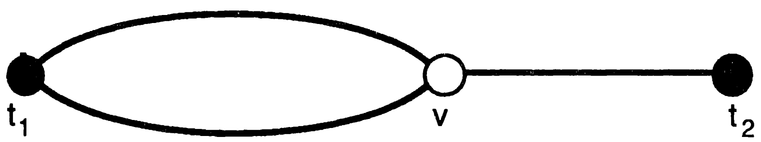
Now, let $A\in \mathcal{A}_1$, and let v be the corresponding node. Since A is a cut set, v must be a "cut node". Thus, since $|A| \leq 3$, it is easily seen that the network corresponding to $(A,\phi_{+(E\backslash A)})$ must be of one of the three types illustrated in Figure 5.2. (t_1, t_2 and t_3 denote the terminals of the systems.) In all cases it is evident that $c(\phi_{+(E\backslash A)}) > 0$. (Indeed $c(\phi_{+(E\backslash A)})=1$ if $(A,\phi_{+(E\backslash A)})$ is of type 1, $c(\phi_{+(E\backslash A)})=2$ if $(A,\phi_{+(E\backslash A)})$ is of type 2, and $c(\phi_{+(E\backslash A)})=3$ if $(A,\phi_{+(E\backslash A)})$ is of type 3.) Hence, (5.5.) is satisfied.

Similarly, it is easily seen that $(A, [\phi_{+(G_A\backslash A)}]_{+C-(F_A\backslash C)})$ must be of the same three types for all $C\in \mathcal{C}_A$ and $A\in \mathcal{A}_2$, (whenever $\mathcal{C}_A$ is non-empty). Hence, $c(\, [\phi_{+(G_A\backslash A)}]_{+C-(F_A\backslash C)}) > 0$ for all $C\in \mathcal{C}_A$ and $A\in \mathcal{A}_2$, (whenever,$\mathcal{C}_A$ is non-empty) implying that (5.6.) is satisfied as well. Thus we conclude that the theorem is true. □

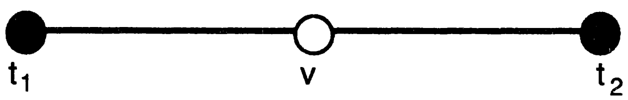
We observe that the above theorem does not apply to cases where some of the external shocks strike sets corresponding to terminal nodes. This is of course a serious restriction. Especially, this implies that if we try to replace unreliable nodes with perfect nodes by performing the network transformation indicated above, then only non-terminal nodes can be treated. However, if a terminal, t, in a k-terminal undirected network system (E,ϕ) is unreliable, then obviously t is in series with the rest of the system. Thus, if θ_t is the reliability of t, then:

(5.7.) $\Pr(\phi=1) = \theta_t \Pr(\phi=1 \mid t \text{ is functioning})$

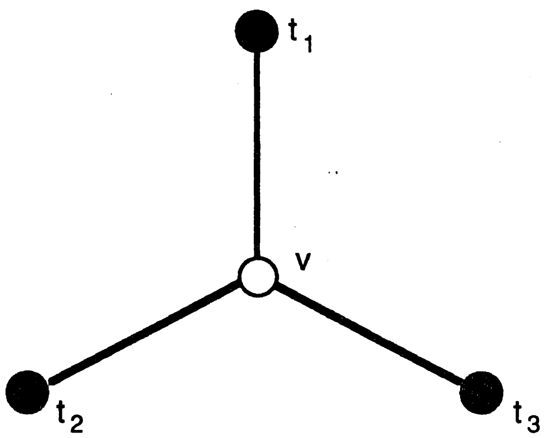
Hence, the reliability of the system may be computed by first computing the reliability of the system considering t as a perfect node, and then multiply this value by θ_t. Thus, unreliable terminals are indeed very easy to handle. It is the unreliable non-terminals that cause the problems.



Type 1-graph.



Type 2-graph.



Type 3-graph.

Figure 5.2. The three possibilities for $(A,\phi_{+(E\setminus A)})$.

Before we move to the second dependence model, we illustrate the above results by an example.

Example 5.2. Consider the network system $(E\cup V,\phi)$ illustrated in Figure 5.3. where $E=\{1,....,6\}$ is the set of edges, $V=\{v,t_1,t_2,t_3\}$ is the set of nodes and $T=\{t_1,t_2,t_3\}$ is the set of terminals. We assume that the edges have reliabilities $\pi_1,\ldots,\pi_6$ respectively while the nodes have reliabilities $\theta,\tau_1,\tau_2,\tau_3$ respectively.

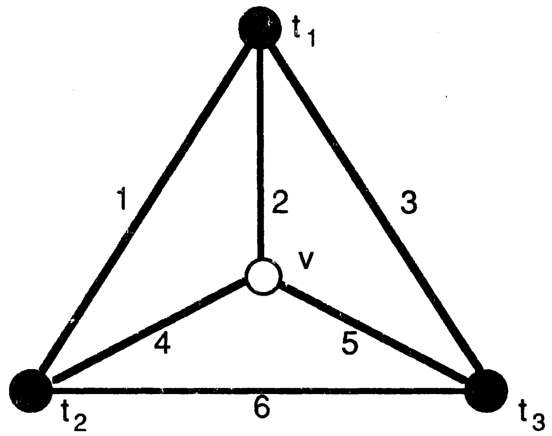


Figure 5.3. A 3-terminal undirected network with unreliable nodes.

We now replace π_i by p_i, i=1,....,6 where $p_1,\ldots,p_6$ are given by:

$$(5.8.) \quad p_i = \begin{cases} \pi_i & , \; i = 1,3,6. \\ \pi_i\,\theta & , \; i = 2,4,5. \end{cases}$$

Furthermore, we replace θ,τ_1,τ_2 and τ_3 by 1. Thus, we have obtained a modified network where all the nodes are perfect. Let R denote the reliability of this network. Since $|E(v)| = |\{2,4,5\}| = 3$, it follows by Theorem 5.1. and (5.7.) that:

$$(5.9.) \quad \Pr(\phi=1) \geq \tau_1\,\tau_2\,\tau_3\,R\;.$$

Hence, by performing the above simplifications, we obtain a conservative estimate of the reliability of the system. Especially, if $\pi_i = \pi$, i=1,...,6, by standard calculations it can be seen that the error is given by:

$$(5.10.) \quad e = \tau_1\,\tau_2\,\tau_3\,\theta\,\pi^2[\,\alpha\theta^2 + \beta\theta + \gamma]$$

where α, β and γ are given by:

$$(5.11.) \quad \alpha = -4\pi^3 + 9\pi^2 - 6\pi + 1 \quad , \quad \beta = 6\pi^2 - 12\pi + 6$$

$$\gamma = 4\pi^3 - 15\pi^2 + 18\pi - 7\,.$$

The maximal error, (occurring when $\tau_1=\tau_2=\tau_3=1$, $\pi=0.53$ $\theta=0.47$) is -0.036 which is not too bad.

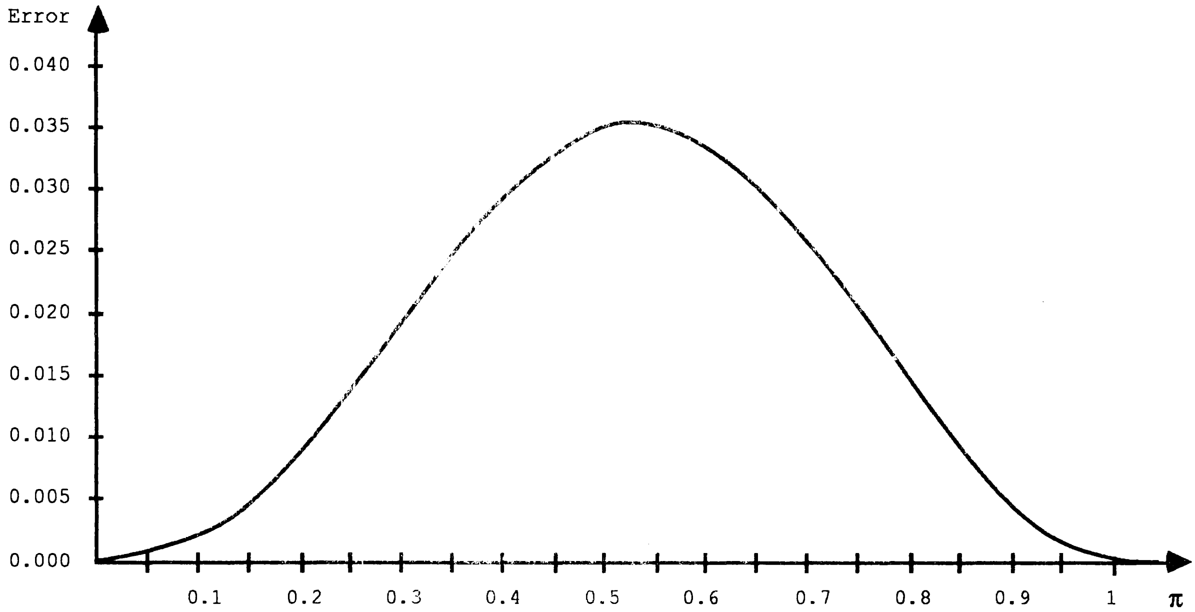


Figure 5.4. The absolute value of the error as a function of the common edge reliability π.

In Figure 5.4. we have plotted the absolute value of the maximal error w.r.t. θ, τ_1, τ_2 and τ_3 as a function of the edge reliability, π.

If $\pi > 0.95$ (as is quite usual in highly reliable systems), then the error is less than 10^{-3}. Thus, the effect of the transformation is indeed quite neglectable. □

We now turn to the dependence model based on stand-by components.

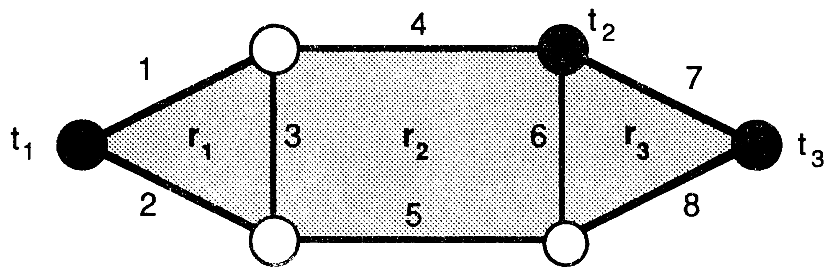


Figure 5.5. A planar network with three regions, r_1, r_2 and r_3.

Model 2. Assume that the dependence between the components of (E,ϕ) is such that a stand-by model is reasonable. We are then again faced with the problem of selecting the family $\mathcal{A}$ of sets corresponding to the external stand-by components. As before, it seems natural to concentrate on sets of components being "close" to each other in the network. One possibility is of course to use the same sets as we did in Model 1. However, in order to obtain a result similar to Theorem 5.1. it appears to be necessary to let $\mathcal{A}$ be a subfamily of $\mathcal{M}$, the family of minimal circuit sets of the network. If f.ex. the network is planar and G is a planar embedding of the network, (i.e a realization of the network in the plane with no edges crossing) a natural choice would be to let $\mathcal{A}$ be the family of circuit sets corresponding to the regions of G. An example of such a network is shown in Figure 5.5. This network contains three regions, denoted by r_1, r_2 and r_3 respectively. The circuit sets corresponding to these are $A_1=\{1,2,3\}$, $A_2=\{3,4,5,6\}$ and $A_3=\{6,7,8\}$.

In general we let $\mathcal{A}=\{A_1,....,A_s\}$ where $|A_j|\geq 2$ and, (as we said), $A_j \in \mathcal{M}$ $j=1,....,s$. For each set $A_j \in \mathcal{A}$ we assume that the corresponding stand-by functions with probability μ_j. Furthermore, for each component $i\in E$ the corresponding internal stand-by functions with probability p_i. Finally, let $\boldsymbol{\mu} = (\mu_1,.....,\mu_s)$ and $\boldsymbol{\pi} = (\pi_1,....,\pi_n)$, and introduce the error function $e(\boldsymbol{\mu},\boldsymbol{\pi})$ = The error caused by neglecting the dependence. (See (4.29.)).

Note that using this model, we may interpret the external stand-by components as components affecting the circuits. More specifically, consider a stand-by j and let c_j be the corresponding circuit. If the stand-by functions, then all the nodes incident to c_j can communicate through the "area inside" c_j.

Thus, proceeding like we did when treating Model 1, we get that neglecting dependence may be viewed as a transformation of systems with "circuit stand-by components" into standard network systems.

Even if network systems with circuit stand-by components might be rare in real life, the above transformation can be useful in order to derive upper bounds on the reliability of a given network system. We shall illustrate this by an example later, but first we provide the basic result on this model. The proof is similar to the proof of Theorem 5.1. and thus omitted.

Theorem 5.3. Consider the k-terminal undirected network system (E,ϕ) described above. Assume now that the system contains only two terminals (i.e. k=2.), and that $|A_j| \leq 3$, $j=1,....,s$. Then $e(\boldsymbol{\mu},\boldsymbol{\pi}) \geq 0$ for all $\mu_1,....,\mu_s, \pi_1,....,\pi_n \in [0,1]$, i.e. the reliability is overestimated. □

We close this section by providing an example where the above theorem can be used.

Example 5.4. (Reversed Δ-Y-reduction.) Rosenthal and Frisque (1977) introduces a network transformation known as the Δ-Y-reduction. The effect of this transformation is shown in Figure 5.6. In many cases this transformation may simplify the reliability calculations. However, this is not always true. It is possible to construct examples where the complexity in fact is increased by performing Δ-Y-reduction. (Especially, the unreliable node produced by the transformation may cause problems.)

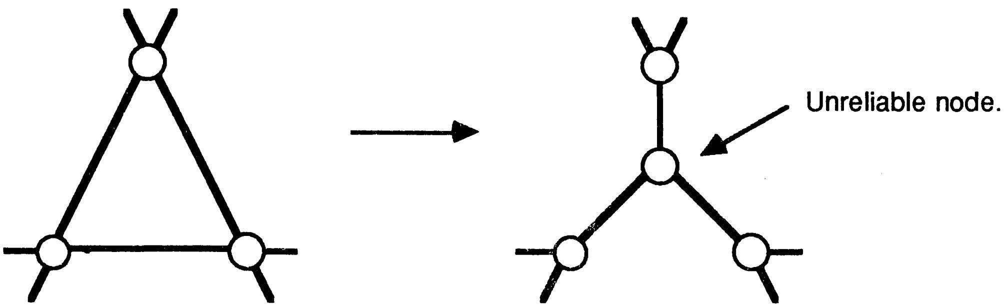


Figure 5.6. Δ-Y-reduction.

In some cases it even turns out that a reversed Δ-Y-reduction would simplify the calculations. However, in order to construct such a transformation in general, it appears to be necessary to introduce a circuit stand-by component. This is done as follows. Assume that we in a network system observes a "Y-configuration" consisting of the components a, b, c and v, as illustrated in the left-hand part of Figure 5.7. It is desired to replace this by the "Δ-configuration" consisting of the components s, t, u and r, shown in the right-hand part of Figure 5.7.

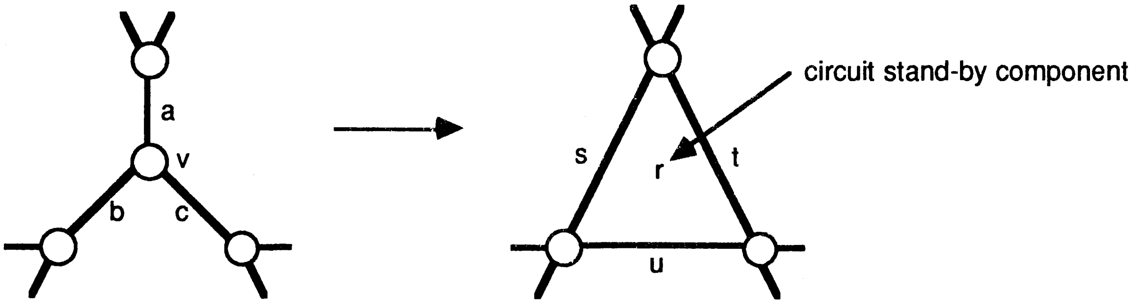


Figure 5.7. Reversed Δ-Y-reduction.

We assume that the components in the original system as well as the transformed system are independent, and that the reliabilities are p_a, p_b, p_c, p_v, p_s, p_t, p_u and p_r respectively. The problem now is to determine p_s, p_t, p_u and p_r such that the reliability of the system is preserved. By component factoring, it is easily seen that the solution to this problem is given by:

(5.12.) $p_s = \alpha/(\alpha+\delta)$, $p_t = \beta/(\beta+\delta)$, $p_u = \gamma/(\gamma+\delta)$ and

$$p_r = 1 - [(\alpha+\delta)(\beta+\delta)(\gamma+\delta)/\delta^2]$$

where α, β, γ and δ are given by:

(5.13.) $\alpha = p_a p_b (1-p_c) p_v$, $\beta = p_a p_c (1-p_b) p_v$, $\gamma = p_b p_c (1-p_a) p_v$

$$\delta = (1-p_v) \amalg [1-(p_a p_b + p_a p_c + p_b p_c - 2 p_a p_b p_c)]$$

Note that using this formula, it may happen that p_r is negative. This implies that it is not always possible to interpret p_r as a probability. However, if we consider the transformation simply as a computational tool, this does not matter as long as the final answer is correct. (A similar phenomenon occurs when using the Δ-Y-reduction, where the unreliable node may have a reliability greater than one. For more details, see Rosenthal and Frisque (1977).)

The problem with this transformation is of course that it produces the circuit stand-by component r. If p_r is negative, we may eliminate the problem simply by deleting this component. By the monotonicity of the reliability function, this will produce an upper bound on the reliability. If on the other hand p_r is positive and the system contains only two terminals, an upper bound on the reliability may be obtained by using Theorem 5.3.

More specifically, we consider the system illustrated in the left-hand part of Figure 5.8. We assume that the nodes u and v both have reliability θ, while the terminals, s and t, are perfect. All the edges have reliability π. Finally, we assume that the components are independent.

By using Theorem 5.1. we may transform the system into a network with perfect nodes, and thus obtain a lower bound on the system reliability by calculating the reliability of the transformed system (f.ex. by using component factoring). This lower bound is given by:

$$(5.14.) \quad l_1(\theta,\pi) = (1-\pi)[\theta^2\pi(2\theta\pi - \theta^2\pi^2)^2 + (1-\theta^2\pi)(2\theta^2\pi^2 - \theta^4\pi^4)] + \pi$$

On the other hand we may perform a reversed Δ-Y-reduction on the Y-configuration consisting of the edges 2, 4 and 5 and the node v. The resulting network is illustrated in the right-hand part of Figure 5.8. The circuit stand-by component r may then be eliminated as described above, yielding an upper bound on the reliability. (Observe that in this case the reversed Δ-Y-reduction is obviously efficient since it transforms a complex network into a simple series-parallel system!) This is given by:

$$(5.15.) \qquad u_1(\theta,\pi) = \pi \amalg p \amalg \{\max(0,p_0)\} \amalg [\theta (\pi \amalg p \amalg \{\max(0,p_0)\})]$$

where p and p_0 are given by:

$$(5.16.) \quad p = \frac{(\theta\pi^2 - \theta\pi^3)}{(1 - 2\theta\pi^2 - \theta\pi^3)} \quad , \quad p_0 = 1 - \frac{(1 - 2\theta\pi^2 - \theta\pi^3)^3}{(1 - 3\theta\pi^2 + 2\theta\pi^3)^2}$$

(Observe that we have to use $\{\max(0,p_0)\}$ instead of just p_0 since it may happen that p_0 is negative.)

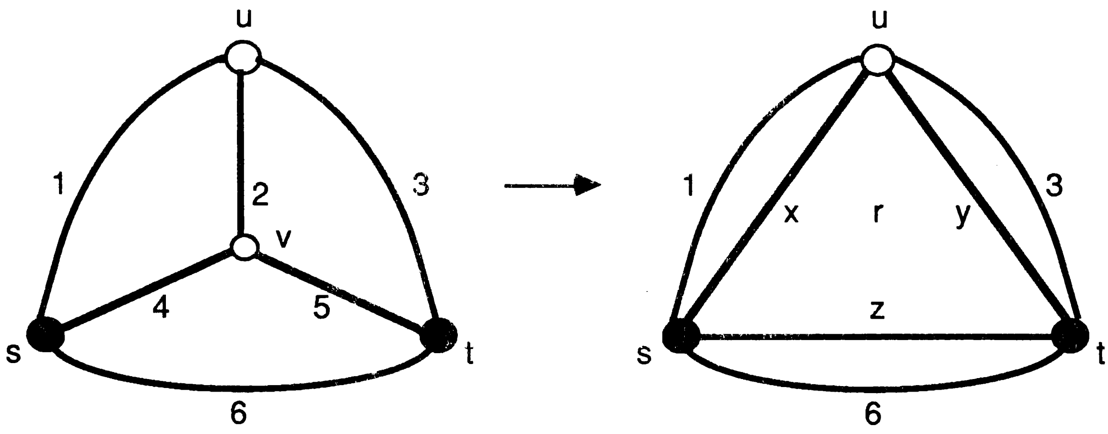


Figure 5.8. Application of the reversed Δ -Y-reduction.

In Table 5.1. $l_1(\theta,\pi)$, $u_1(\theta,\pi)$ as well as the correct value $h(\theta,\pi)$ are given for π=0.5 and different values of θ. In addition we have tabled some other more well-known bounds on the reliability. Specifically, we have:

$$(5.17.)\quad l_2(\theta,\pi) = \prod_{j=1}^{k} \coprod_{i\in K_j} p_i \quad , \quad u_2(\theta,\pi) = \coprod_{j=1}^{p} \prod_{i\in P_j} p_i$$

$$(5.18.)\quad l_3(\theta,\pi) = \max_{1\le j\le p} \prod_{i\in P_j} p_i \quad , \quad u_3(\theta,\pi) = \min_{1\le j\le k} \coprod_{i\in K_j} p_i$$

where $P_1,\ldots\ldots,P_p$ are the minimal path sets of the system, $K_1,\ldots\ldots,K_k$ are the minimal cut sets of the system, and the p_i-s are the component reliabilities (i.e. $p_i = \theta$ if the component i is a node, and $p_i = \pi$ if the component i is an edge). (For more details see Barlow and Proschan (1981).)

θ	$l_3(\theta,\pi)$	$l_2(\theta,\pi)$	$l_1(\theta,\pi)$	$h(\theta,\pi)$	$u_1(\theta,\pi)$	$u_2(\theta,\pi)$	$u_3(\theta,\pi)$
0.00	0.50000	0.10645	0.50000	0.50000	0.50000	0.50000	0.50000
0.10	0.50000	0.14443	0.50251	0.52492	0.52522	0.52587	0.59500
0.20	0.50000	0.18742	0.51011	0.54972	0.55082	0.55325	0.68000
0.30	0.50000	0.23535	0.52297	0.57442	0.57670	0.58175	0.75500
0.40	0.50000	0.28803	0.54124	0.59906	0.60276	0.61103	0.82000
0.50	0.50000	0.34514	0.56494	0.62369	0.62887	0.64073	0.87500
0.60	0.50000	0.40617	0.59388	0.64839	0.65490	0.67053	0.87500
0.70	0.50000	0.47046	0.62767	0.67323	0.68068	0.70009	0.87500
0.80	0.50000	0.53712	0.66563	0.69833	0.70602	0.72915	0.87500
0.90	0.50000	0.60506	0.70680	0.72384	0.73070	0.75742	0.87500
1.00	0.50000	0.67291	0.75000	0.75000	0.75442	0.78466	0.87500

Table 5.1. Bounds on the reliability for different values of θ, (π=0.5.).

As is seen from the table, our bounds are superior to the traditional bounds for all values of θ. However, this will not be true for all types of systems. Since the precision decreases for each non-excact transformation we perform, our bounds will be outperformed if the number of unreliable non-terminal nodes and the number of circuit stand-by components are high. Still the main advantage with our bounds is that they can be computed without knowing the minimal path and cut sets.

6. Conclusions.

A typical problem when computing the reliability of a system of dependent components, is the lack of information on the joint distribution of the component states. Often the available information (if any at all) is more qualitative than quantitative. That is, one may know something about the underlying structure or the sources of the dependence. Still, the information is insufficient in order to specify a complete distribution.

In this paper we have focused on situations where, apart from the marginal component reliabilities, only the basic structure of the joint distribution is known. A main conclusion is that at least in some cases this type of information can be used to determine the sign of the possible error in the reliability if the dependence is neglected.

By using methods similar to ours it is possible to develop results where the restrictions on the system are weaker. However, in order to use these results, more information concerning the joint distribution has to be known.

F.ex. it can be shown that if a BMS, (E,ϕ) does not contain cut sets of cardinality one, then there exists an $\varepsilon>0$ $(\varepsilon\leq 1)$ and a $\delta(\theta_E)>0$ $(\delta(\theta_E)\leq 1)$ such that the reliability is overestimated if $\theta_E \in [1-\varepsilon,1]$ and $\theta_A \in [1-\delta(\theta_E),1]$ for all $\varnothing\subset A\subset E$. Using this formulation we obtain a considerably weaker cut structure condition than the one given in Theorem 4.4. However, in this case the interval for the θ_A-s $(A\subset E)$ depends on θ_E. Thus, it is necessary to know this parameter to obtain a conclusion. (If θ_E and the marginal component reliabilities are known, a conclusion may be obtained by using a similar argument as we did in (4.26.).)

We have chosen to concentrate on cases where as little as possible is known concerning the joint distribution, because we believe that this is perhaps the most common situation. However, the presentation is meant to illustrate methods which can be extended to other situations as well. One possible application of our results is to use them in order to identify important parameters of the model, estimate these and then obtain a conclusion.

More or less as a biproduct of our study, we have demonstrated that our methods may be used in order to obtain bounds on the reliability of network systems. The potential possibilities of these methods is not at all covered in this paper. Our intention so far has only been to present the basic ideas. We hope to return to this subject in a later paper.

7. References.

Barlow, R.E. and Iyer, S. (1985). "Computational Complexity of Coherent Systems and the Reliability Polynomial." ORC 85-6. Operations Research Center, Univ. of Calif., Berkeley.

Barlow, R.E. and Proschan, F. (1981). "Statistical Theory of Reliability and Life Testing." To Begin with, Silver Spring, MD.

Boyles, R.A and Samaniego, F.J. (1984). "Modeling and Inference for Multivariate Binary Data With Positive Dependence." J. American Stat. Assoc. **79** 188-193.

Egeland, T. (1985). "Dependence and its Consequences for Reliability Calculations." Unpublished Cand. Scient. Thesis, Univ. of Oslo.

Hagstrom, J. (1986). "The Reliability Function of Binary Systems: A New Perspective on the Domination Results." Working Paper 86-12. College of Business Administration, Univ. of Illinois at Chicago.

Hagstrom, J. and Mak, K.T. (1986). "System Reliability Analysis in the Presence of Correlated Component Failures." Working Paper 86-11. College of Business Administration, Univ. of Illinois at Chicago.

Huseby, A.B. (1984). "A Unified Theory of Domination and Signed Domination with Application to Exact Reliability Computations." Stat. Res. Report No. 3, Univ. of Oslo.

Huseby, A.B. (1985). "Domination Theory and the Crapo ß-invariant." Submitted to Networks.

Huseby, A.B. (1986). "A Representation of Multivariate Binary Variables with Application to Reliability Theory." Res. Report. Center for Industrial Research.

Marshall, A. and Olkin, I. (1967). "A Multivariate Exponential Distribution." J. American Stat. Assoc. **62** 30-44.

Rosenthal, A. and Frisque, P. (1977). "Transformations for Simplifying Network Reliability Calculations." Networks. **7** 97-111.

Satyanarayana, A. (1982). "A Unified Formula for Analysis of Some Network Reliability Problems." IEEE Trans. on Reliability. R-31, 23-32.

Satyanarayana, A. and Chang, M.K. (1983). "Network Reliability and the Factoring Theorem." Networks. **13** 107-120.

Satyanarayana, A. and Prabhakar, A. (1978). "New Topological Formula and Rapid Algorithm for Reliability Analysis of Complex Networks." IEEE Trans. on Reliability. R-27, 82-100.

Wood, R.K. (1985). "A Factoring Algorithm Using Polygon-to-Chain Reductions for computing K-Terminal Network Reliability." Networks. **15** 173-190.